Gravitational instantons and internal dimensions.

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We study instanton solutions in general relativity with a scalar field. The metric ansatz we use is composed of a particular warp product of general Einstein metrics, such as those found in a number of cosmological settings, including string cosmology, supergravity compactifications and general Kaluza Klein reductions. Using the Hartle-Hawking prescription the instantons we obtain determine whether metrics involving extra compact dimensions of this type are favoured as initial conditions for the universe. Specifically, we find that these product metric instantons, viewed as constrained instantons, do have a local minima in the action. These minima are then compared with the higher dimensional version of the Hawking-Turok instantons, and we argue that the latter always have lower action than those associated with these product metrics.

I. INTRODUCTION

Consistent unified field theories which include gravity appear to indicate that the Universe has more than four spacetime dimensions. An obvious problem which follows is how to interpret these unseen extra dimensions? One approach that has been followed is to postulate that only four of these are observable, the extra dimensions have managed to become compact and are unobservably small. Recently however there has been a tremendous amount of interest in the effective five dimensional cosmologies associated with Branes, in which the fifth dimension can be macroscopic in size, yet remain unobservable at low energies. In general, these compactified spaces are assumed as part of the initial metric ansatz, and the cosmology of such metrics is then determined. Although this is a natural approach to take, it does not address the issue of whether such an initial condition is to be expected in string or M -theory, for example. Is there any way in which we can calculate the probability of the Universe possessing such compact internal dimensions as an initial condition? It would be of great interest if it could be shown that quantum cosmology predicts a manifold with compact extra dimensions as the most likely initial configuration.

Symmetry arguments usually provide a very powerful tool for determining which instanton solutions should provide the dominant contribution (i.e. those with lowest Euclidean action) to the Hartle Hawking path integral [1], hence providing the most likely background spacetime. An example is the Hawking-Moss instanton, involving a scalar field ϕ with potential $V(\phi)$ [2]. Assuming the potential had a stationary point at some non-zero value they obtained in four spacetime dimensions an O(5) symmetric instanton solution where ϕ is constant and the Euclidean manifold is a four sphere.

However, Coleman and De Lucia [3] obtained an instanton solution of lower action with O(4) symmetry which was non-singular and corresponded to the nucleation of a bubble of true vacuum in a sea of false vacuum deSitter space. It was used in the earliest versions of open inflation [4], because the interior of such a bubble is in fact an open universe. Hawking and Turok [5] took these solutions one step further, dropping the requirement for non-singular instanton configurations; they obtained solutions where the scalar field potential increased monotonically from a single minimum. These solutions also allowed for a natural continuation to an open universe which was inflating. Moreover, although the instanton solutions themselves were singular their action was finite. Indeed they demonstrated a family of solutions which had lower action than the more symmetric O(5) solution! The notion that the O(4) symmetry of the Hawking-Turok instanton was responsible for the low action was tested in [6]. Treating the instanton as a foliation of squashed rather than round three spheres, it was found that the O(4) instanton was the lowest action solution within this family. In an interesting paper Garriga [7] proposed a resolution to the problem of having a singularity in the solution; singular instantons can arise from compactifications of regular higher dimensional instantons when viewed as lower dimensional objects.

In this paper we investigate the nature of instanton solutions for the largest range of cosmologically relevant higher dimensional metrics that have been studied to date. Our results will be of relevance for the study of any higher

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dimensional model which involves compactifications on Einstein metrics, i.e. models of string cosmology involving compactifications on tori, supergravity compactifications on spheres and string theories where the compactified dimensions are Calabi-Yau manifolds. In particular we will be investigating instanton solutions arising from the metric ansatz,

$$\mathrm{ds}^2 = \mathrm{d}\xi^2 + \mathrm{a}^2_{(1)}(\xi)\mathrm{ds}^2_{(1)} + \mathrm{a}^2_{(2)}(\xi)\mathrm{ds}^2_{(2)} + ... + \mathrm{a}^2_{(\mathrm{T})}(\xi)\mathrm{ds}^2_{(\mathrm{T})}. \tag{1}$$

The only restriction on the $ds^2_{(i)}$'s is that they are Einstein metrics on compact manifolds; the Ricci tensor is proportional to the metric. Of the many solutions that exist, we will see how a class of these instantons may be continued to a four dimensional inflating universe, with a number of static extra dimensions.

In general, because of the non-linear nature of the equations, the solutions for the scale factors a_i are obtained numerically, and from these we can study the action of the (generically singular) instantons. The most important result we obtain is that the family of singular instantons of this type can provide a local minima of the action for non trivial extra dimensions. However, it turns out that in all the cases we examined the action of these local minima remains *higher* than that of the corresponding higher dimensional Hawking-Turok instanton. The implication of such a result is important. The symmetry properties associated with the Hawking-Turok instanton appear to determine the most likely instanton configuration, at least for the cases involving Einstein metrics.

The layout of the rest of the paper is as follows: In section II we derive the field equations and action associated with our metric. Section III contains the results of our numerical and analytical analysis and presents the nature of the local minima of the action. It also contains the comparison of these instantons with the equivalent higher dimensional Hawking-Turok case and shows how the latter always lead to a lower Euclidean action. Section IV presents exact solutions for the case of a cosmological constant replacing the scalar field potential. We also mention the analytical continuation of our solutions to a space-time with a lorentzian signature and demonstrate the existence of solutions where the internal dimensions remain static while the four dimensional spacetime is inflating. We draw our conclusions in section V.

II. DERIVATION OF FIELD EQUATIONS

Our starting point is a manifold $\mathcal{M}$ which has a metric structure imposed on it, and a scalar field ϕ living on it with potential $\mathcal{V}(\phi)$. By using the usual torsion free, metric connection on $\mathcal{M}$ we can describe the equations of motion for the metric and for ϕ which follow from the Einstein-Hilbert action.

$$S_{\mathrm{E}} = \int_{\mathcal{M}} \eta \left[-\frac{1}{2\kappa^2}\mathcal{R} + \frac{1}{2}(\partial\phi)^2 + \mathcal{V}(\phi) \right] + \text{boundary term}. \tag{2}$$

Here, $\kappa^2 = 8\pi/m^2_{\mathrm{Pl}}$ (scaled to unity for the rest of the paper), and the boundary terms are such that the action does not contain second derivatives of the metric [8] [9]. η is the volume form and $\mathcal{R}$ is the Ricci scalar of the connection.

As mentioned earlier we consider the manifold $\mathcal{M}$ as a foliation in Euclidean time of a product of boundary-less manifolds. At any given time ξ we can then write $\mathcal{M}(\xi)$ as a Cartesian product of $\mathcal{M}_{(i)}$, $i = 1...T$ each with dimension $n_{(i)}$, where for convenience we define $N = n_{(1)} + n_{(2)} + ... + n_{(T)}$. To endow $\mathcal{M}$ with a metric structure we start by putting a metric on each of the $\mathcal{M}_{(i)}$, denoted $\mathrm{ds}^2_{(\mathrm{i})}$. The metric structure we impose on $\mathcal{M}$ then follows by introducing a ξ dependent scale factor, $a_{(i)}$, for each $\mathcal{M}_{(i)}$; providing information about the relative size of the $\mathcal{M}_{(i)}$ at any given ξ. The resulting metric is then given by Eq. (1).

This form for the metric is very general. It includes a wide class of metrics commonly considered in cosmology, such as those leading to the Coleman-De Lucia instanton [3], Kantowski-Sachs instantons [10], Hawking-Turok instanton [11], most of the models of string cosmology arising out of compactifications on tori (for a review see [12]), compactifications of string theory on Calabi-Yau spaces (for a review see [13]), and some compactifications of Supergravity theories on spheres (for a review see [14]). For example in [3] [11] $T = 1$ and $\mathcal{M}_{(1)}$ is a three sphere with its standard round metric. A more exotic Kantowski-Sachs metric was considered in [10], there $T = 2$ with $\mathcal{M}_{(1)} = S^1$, $\mathcal{M}_{(2)} = S^2$.

The equations of motion for the scale factors are derived using the Einstein-Hilbert action, for which we need to calculate the components of the Riemann tensor. This is made simpler by using methods from differential geometry [15], so we start by defining an orthonormal basis of one forms on $\mathcal{M}$.

$$\begin{aligned} \omega^0 &= \mathrm{d}\xi \\ \omega^{\bar{\mu}}_{(i)} &= a_{(i)}\bar{\omega}^{\bar{\mu}}_{(i)} \end{aligned} \tag{3}$$

The notation we are using is that barred quantities correspond to properties on the submanifolds $\mathcal{M}_{(i)}$. So, in (3) the $\bar{\omega}^{\bar{\mu}}_{(i)}$ are an orthonormal basis of one forms with respect to the metric $\mathrm{ds}^2_{(\mathrm{i})}$ and $\bar{\mu} = 1...n_{(i)}$, whereas the $\omega^{\bar{\mu}}_{(i)}$ are in the orthonormal basis of ds^2. The notation for the orthonormal basis of ds^2 (ω^μ) is such that $\omega^\mu = \omega^{\bar{\mu}}_{(i)}$, $\mu = 1...N$. So because $\bar{\mu} = 1...n_{(i)}$ we find $\bar{\mu} = \mu - (n_{(1)} + n_{(2)} + ... + n_{(i-1)})$. This should be unambiguous (although it might not seem so at first glance!) as a barred index always appears on a quantity with a subscript (i) saying which $\mathcal{M}_{(i)}$ it lives on.

The connection one forms on the $\mathcal{M}_{(i)}$ ($\bar{\Theta}^{\bar{\mu}}_{(i)\bar{\nu}}$) are taken to be torsion free metric connections,

$$\mathrm{d}\bar{\omega}^{\bar{\mu}}_{(\mathrm{i})} + \bar{\Theta}^{\bar{\mu}}_{(\mathrm{i})\bar{\nu}} \wedge \bar{\omega}^{\bar{\nu}}_{(\mathrm{i})} = 0 \tag{4}$$

$$\bar{\Theta}_{(i)\bar{\nu}\bar{\mu}} = -\bar{\Theta}_{(i)\bar{\mu}\bar{\nu}} \qquad \bar{\mu}, \bar{\nu} = 1...n_{(i)}.$$

The connection forms on $\mathcal{M}$ satisfy similar relations

$$\mathrm{d}\omega^\mu + \Theta^\mu{}_\nu \wedge \omega^\nu = 0 \tag{5}$$

$$\Theta_{\nu\mu} = -\Theta_{\mu\nu} \qquad \mu, \nu = 0...N.$$

To evaluate the connection one forms we use (3) to find

$$\mathrm{d}\omega^0 = 0 \tag{6}$$

$$\mathrm{d}\omega^{\bar{\mu}}_{(\mathrm{i})} = \alpha'_{(i)}\omega^0 \wedge \bar{\omega}^{\bar{\mu}}_{(i)} + a_{(i)}\mathrm{d}\bar{\omega}^{\bar{\mu}}_{(\mathrm{i})}, \tag{7}$$

where we have introduced $\alpha_{(i)} = \ln(a_{(i)})$ and the prime denotes the derivative with respect to ξ. Taking the definition of $\Theta^\mu{}_\nu$ in (5) and using (4) for the individual $\mathcal{M}_{(i)}$ we find

$$\Theta^0_{(i)\bar{\mu}} = -\alpha'_{(i)}\omega^{\bar{\mu}}_{(i)} \tag{8}$$

$$\Theta^{\bar{\mu}}_{(i)\bar{\nu}} = \bar{\Theta}^{\bar{\mu}}_{(i)\bar{\nu}} \qquad \bar{\mu}, \bar{\nu} = 1...n_{(i)}. \tag{9}$$

So, if the indices on $\Theta^\mu{}_\nu$ correspond to different $\mathcal{M}_{(i)}$ then that connection form vanishes. We must also take care to note that $\bar{\Theta}^{\bar{\mu}}_{(i)\bar{\nu}}$ is defined using the basis on $\mathcal{M}_{(i)}$ ($\bar{\omega}^{\bar{\mu}}_{(i)}$) whereas $\Theta^{\bar{\mu}}_{(i)\bar{\nu}}$ uses that on $\mathcal{M}$ ($\omega^{\bar{\mu}}_{(i)}$).

Now that the connection forms on $\mathcal{M}$ are known, in terms of those on $\mathcal{M}_{(i)}$, we may calculate the curvature two forms [15],

$$R^\mu_\nu = \mathrm{d}\Theta^\mu_\nu + \Theta^\mu_\rho \wedge \Theta^\rho{}_\nu \qquad \mu, \nu, \rho = 0...\mathrm{N}. \tag{10}$$

There is an analogous expression for the curvatures on $\mathcal{M}_{(i)}$, where the appropriate barred connections are used. Using (4) one finds for the curvature forms on $\mathcal{M}$.

$$R^0_{(i)\bar{\mu}} = -[\alpha''_{(i)} + (\alpha'_{(i)})^2]\omega^0 \wedge \omega^{\bar{\mu}}_{(i)} \tag{11}$$

$$R^{\bar{\mu}}_{(i)\bar{\nu}} = \bar{R}^{\bar{\mu}}_{(i)\bar{\nu}} - (\alpha'_{(i)})^2\omega^{\bar{\mu}}_{(i)} \wedge \omega^{\bar{\nu}}_{(i)}$$

$$R^{\bar{\mu}}_{(i,j)\bar{\bar{\nu}}} = -\alpha'_{(i)}\alpha'_{(j)}\omega^{\bar{\mu}}_{(i)} \wedge \omega^{\bar{\bar{\nu}}}_{(j)}$$

The notation for the last equation of (11) is that $i \neq j$ and the single barred index corresponds to $\mathcal{M}_{(i)}$ with the double barred index living on $\mathcal{M}_{(j)}$. Again we stress that $\bar{R}^{\bar{\mu}}_{(i)\bar{\nu}}$ is defined with the $\bar{\omega}^{\bar{\mu}}_{(i)}$ basis, which differs from the basis on $\mathcal{M}$ by a factor of $a_{(i)}(\xi)$.

Einstein's equations relate the Ricci tensor to the stress-energy tensor. For the above curvature two forms we use $R^\mu_\nu = \frac{1}{2}\mathcal{R}^\mu{}_{\nu\rho\sigma}\omega^\rho \wedge \omega^\sigma$, enabling us to calculate the Ricci tensor $\mathcal{R}_{\mu\nu} = \mathcal{R}^\rho{}_{\mu\rho\nu}$ and Ricci scalar $\mathcal{R} = \mathcal{R}^\mu_\mu$:

$$\mathcal{R}_{00} = -n_{(1)}[\alpha''_{(1)} + (\alpha'_{(1)})^2] - n_{(2)}[\alpha''_{(2)} + (\alpha'_{(2)})^2]... \tag{12}$$

$$\mathcal{R}^{(i)}_{\bar{\mu}\bar{\nu}} = \frac{1}{a^2_{(i)}}\bar{\mathcal{R}}^{(i)}_{\bar{\mu}\bar{\nu}} \quad (\mu \neq \nu) \tag{13}$$

$$\mathcal{R}^{(i)}_{\bar{\mu}\bar{\mu}} = -\alpha''_{(i)} + \frac{1}{a^2_{(i)}}\bar{\mathcal{R}}^{(i)}_{\bar{\mu}\bar{\mu}} - \alpha'_{(i)}[n_{(1)}\alpha'_{(1)} + n_{(2)}\alpha'_{(2)} + ...] \tag{14}$$

$$\mathcal{R} = -2\left(n_{(1)}\alpha''_{(1)} + n_{(2)}\alpha''_{(2)} + ...\right) - \left(n_{(1)}(\alpha'_{(1)})^2 + n_{(2)}(\alpha'_{(2)})^2 + ...\right) \tag{15}$$

$$- \left(n_{(1)}\alpha'_{(1)} + n_{(2)}\alpha'_{(2)} + ...\right)^2 + \left(\frac{1}{a^2_{(1)}}\bar{\mathcal{R}}^{(1)} + \frac{1}{a^2_{(2)}}\bar{\mathcal{R}}^{(2)} + ...\right),$$

where the repeated $\bar{\mu}$ index in (14) is not summed over. The Einstein tensor is defined by $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}$, which with the Einstein-Hilbert action (2) leads to

$$G_{00} = \frac{1}{2}\phi'^2 - \mathcal{V}(\phi) \tag{16}$$

$$G^{(i)}_{\bar{\mu}\bar{\nu}} = 0 \tag{17}$$

$$G^{(i)}_{\bar{\mu}\bar{\mu}} = -\frac{1}{2}\phi'^2 - \mathcal{V}(\phi) \quad \text{(no sum)}. \tag{18}$$

The key breakthrough now is to realise that for (17) and (18) to be consistent then $\bar{\mathcal{R}}^{(i)}_{\bar{\mu}\bar{\nu}}$ must be constants for $\bar{\mu} = \bar{\nu}$ and vanish otherwise. As we are using an orthonormal basis, this is precisely the statement that the metrics $\mathrm{ds}^2_{(\mathrm{i})}$ are Einstein metrics. In fact the equations are independent of what that metric is because the only effect a different Einstein metric has is to change the constant of proportionality between the Ricci and metric tensor, which may then be absorbed into the scale factors (if it is non-vanishing). This means we may replace $\bar{\mathcal{R}}^{(i)}_{\bar{\mu}\bar{\mu}}$ in (14) by $\Lambda_{(i)} = 0, \pm 1$ and $\bar{\mathcal{R}}^{(i)}$ in (15) by $n_{(i)}\Lambda_{(i)}$ without loss of generality, as long as we remember to rescale the action appropriately.

We may also see what boundary terms are required in (2) by integrating the Ricci scalar by parts to find the boundary contribution. The volume form on $\mathcal{M}$ is given by the wedge product of the volume forms on the $\mathcal{M}_{(i)}$,

$$\eta = a_{(1)}^{n_{(1)}} a_{(2)}^{n_{(2)}} ... a_{(T)}^{n_{(T)}} \omega^0 \wedge \eta_{(1)} \wedge \eta_{(2)} ... \wedge \eta_{(T)}. \tag{19}$$

By defining $V_{(i)}$ as the volume of $\mathcal{M}_{(i)}$ and β to be the product of the scale factors we find that the action, including boundary terms, is given by

$$S_{\mathrm{E}} = V_{(1)}V_{(2)}...V_{(T)} \left\{ -\left[\frac{\partial\beta}{\partial\xi}\right]_{\xi_S}^{\xi_N} + \int \mathrm{d}\xi\beta \left[-\frac{1}{2}\mathcal{R} + \frac{1}{2}\phi'^2 + \mathcal{V}(\phi)\right]\right\} \tag{20}$$

$$= V_{(1)}V_{(2)}...V_{(T)} \left\{ -\left[\frac{\partial\beta}{\partial\xi}\right]_{\xi_S}^{\xi_N} - \frac{2}{n_{(1)} + n_{(2)} + ... + n_{(T)} - 1} \int \mathrm{d}\xi\beta(\xi)\mathcal{V}(\phi)\right\} \tag{21}$$

$$\beta(\xi) = a_{(1)}^{n_{(1)}} a_{(2)}^{n_{(2)}} ... a_{(T)}^{n_{(T)}}. \tag{22}$$

In arriving at the second equation we have used the trace of Einstein's equations (16)-(18) to eliminate the scalar curvature and scalar field kinetic terms. The quantities ξ_S and ξ_N refer to the range of the ξ coordinate, with ξ_N being the 'north' pole and ξ_S referring to the 'south' pole of the instanton taken to be $\xi = 0$. To save writing out the volumes of all the submanifolds we shall call the term in the curly braces of (21) the *reduced action*.

The preceding calculation shows that for metric (1) to be consistent then the metrics on the $\mathcal{M}_{(i)}$ must be Einstein. Given this, the evolution equations for the scale factors $a_{(i)}(\xi)$ depend only on the value of the 'cosmological constants' $\Lambda_{(i)}$ and not on the detailed topology or geometry of the manifold. This is potentially very significant, for any statements we can make about the evolution of the scale factors cover a very large class of manifolds, all those admitting an Einstein metric. Whilst there are manifolds which do not admit an Einstein metric due to topological restriction, there is a large range which do. For example, all semi-simple Lie groups have a Killing metric which is Einstein, along with a large class of quotient spaces. It is noted that any given manifold may have more than one Einstein metric, [16].

III. NUMERICAL SOLUTIONS AND ACTIONS FOR INSTANTON CONFIGURATIONS.

To make some specific predictions we shall in this section numerically investigate the case where there are just two submanifolds $\mathcal{M}_a$, $\mathcal{M}_b$ with scale factors $a(\xi)$ and $b(\xi)$ respectively. The dimension of and 'cosmological constant' associated with these manifolds are taken as n_a, n_b and Λ_a, Λ_b respectively. The equations of motion are then,

$$\frac{1}{2}n_a(n_a-1)\alpha'^2 + \frac{1}{2}n_b(n_b-1)\beta'^2 + n_a n_b\alpha'\beta' - \frac{1}{2}\frac{n_a\Lambda_a}{a^2} - \frac{1}{2}\frac{n_b\Lambda_b}{b^2} = \frac{1}{2}\phi'^2 - \mathcal{V} \tag{23}$$

$$(n_a-1)\frac{a''}{a} + n_b\frac{b''}{b} + \frac{1}{2}(n_a-2)\left[(n_a-1)\alpha'^2 - \frac{\Lambda_a}{a^2}\right] + \frac{1}{2}n_b\left[(n_b-1)\beta'^2 - \frac{\Lambda_b}{b^2}\right] + n_b(n_a-1)\alpha'\beta' = -\frac{1}{2}\phi'^2 - \mathcal{V} \tag{24}$$

$$n_a\frac{a''}{a} + (n_b-1)\frac{b''}{b} + \frac{1}{2}n_a\left[(n_a-1)\alpha'^2 - \frac{\Lambda_a}{a^2}\right] + \frac{1}{2}(n_b-2)\left[(n_b-1)\beta'^2 - \frac{\Lambda_b}{b^2}\right] + n_a(n_b-1)\alpha'\beta' = -\frac{1}{2}\phi'^2 - \mathcal{V} \tag{25}$$

$$\phi'' + (n_a\alpha' + n_b\beta')\phi' = \frac{\partial\mathcal{V}}{\partial\phi} \tag{26}$$

In solving these equations, we used a simple potential, namely $V(\phi) = \frac{1}{2}\phi^2$, although our results do not qualitatively depend on the exact shape of the potential. The main solutions of interest here are the cases where the south pole is regular and there is a curvature singularity at the north pole [5]. Other cases where both the north and south poles are singular have been studied [17]. We shall not concentrate on these cases here as the interesting features we wish to discuss are found in the case where only the north pole is singular.

As we want the south pole to be a smooth endpoint this places conditions on the scale factors. In order to avoid a conical curvature singularity then only one scale factor may vanish there, with all others approaching a constant. We order the $\mathcal{M}_{(i)}$ such that $a_{(1)}$ vanishes at $\xi = 0$, according to

$$a_{(1)}(\xi \to 0) \to \sqrt{\frac{\Lambda_{(i)}}{n_i - 1}}\ \xi, \tag{27}$$

We see then that we must have $\Lambda_{(1)} > 0$ for everything to be well defined and for the solution to be non-trivial.

Before we proceed we need to know what effect the singularity is going to have on the action; in particular, does it remain finite? To decide this we make the assumption that at the singularity the potential is not important, although there are exceptions when exponential potentials are used [18]. We may then integrate (26) to obtain

$$\phi'(\xi \to \xi_N) \propto (a^{n_a} b^{n_b})^{-1}. \tag{28}$$

Now assume a polynomial behaviour for the scale factors near the singularity of the form

$$a(\xi \to \xi_N) \propto (\xi - \xi_N)^p \qquad b(\xi \to \xi_N) \propto (\xi - \xi_N)^q. \tag{29}$$

This is consistent with (23-26) so long as $(q, p \leq 1)$. Then the dominant behaviour on the left hand side of (23) is $(\xi - \xi_N)^{-2}$, giving $n_a p + n_b q = 1$. The volume factor, $\beta = a^{n_a} b^{n_b}$ (21), then goes linearly to zero at ξ_N. Moreover, as ϕ' is diverging as $(\xi - \xi_N)^{-1}$, ϕ diverges logarithmically which is slow enough that the linear volume factor renders the singularity integrable for our potential.

We include some representative results below for the case of two Einstein metrics of dimensions $n_{(1)} = 3$ and $n_{(2)} = 2$. To be specific we have chosen to take the value of $\Lambda_{(i)} = n_{(i)} - 1$, which is the appropriate value for the round metric on $S^{n_{(i)}}$. We shall explain the reason for this shortly.

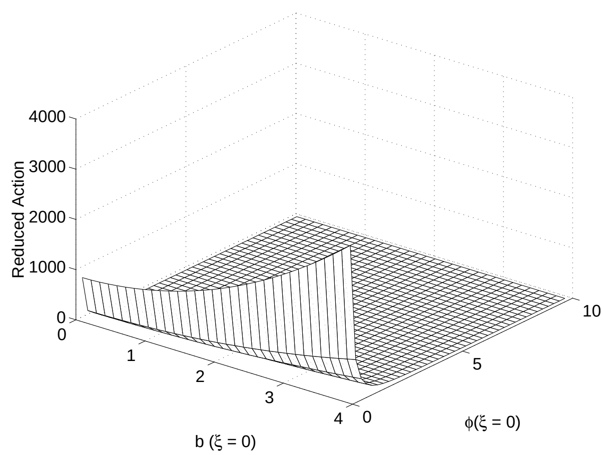

FIG. 1. Plot of the reduced action for case $n_{(a)} = 3, n_{(b)} = 2$

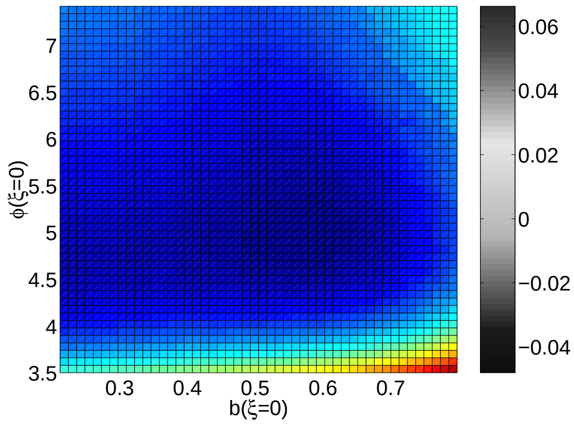

FIG. 2. Detail of the reduced action plot showing a minima for $n_{(a)} = 3, n_{(b)} = 2$

The second of the plots above is a magnification of some structure on the first, showing a minima with negative reduced action. So, just as the Hawking-Turok instanton had a minima in the action so does this new solution which has non-trivial 'spatial' topology. The issue we need to address now is which has a lower action and is therefore more likely as initial conditions. For this we need to know the full Euclidean action rather than the reduced action, this means that the volume prefactors in (21) must be found. Naively it may seem that the total action may be made arbitrarily large and negative by choosing the Einstein metrics such that the volume of the manifolds is arbitrarily large. However, a result from Riemannian geometry, Bishop's theorem [16] implies that for positive Λ the metric which maximises the volume of the manifold is the round sphere. So by looking at the volumes appropriate for the round metric we are looking at the lowest possible action for *all* $\Lambda_{(i)} > 0$ instantons. The volume of S^n with the round metric is

$$\mathrm{V}ol(\mathrm{S^n}) = 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right), \tag{30}$$

so for our example of $n_{(1)} = 3$, $n_{(2)} = 2$ the total action is $S_{\mathrm{E}} = (2\pi^2)(4\pi)(-0.048) = -11.9$, where the reduced action of -0.048 follows from the minima of Fig. 2. We now need to see how this compares to the Hawking-Turok solution in higher dimensions. The starting point is the metric ansatz. This corresponds to (1) with $T = 1$ and $\mathrm{ds}^2_{(1)}$ the round metric on S^n, which gives the equations

$$\frac{a''}{a} = -\left(2\frac{\mathcal{V}(\phi)}{n(n-1)} + \frac{\phi'^2}{n}\right) \tag{31}$$

$$\phi'' + n\phi'\frac{a'}{a} = \mathcal{V}'(\phi). \tag{32}$$

The action is:

$$S_E = -\mathrm{V}ol(\mathrm{S^n})\left((\mathrm{a}(\xi_\mathrm{N})^\mathrm{n})' + \frac{2}{\mathrm{n}-1}\int \mathrm{d}\xi(\mathrm{a}(\xi)^\mathrm{n}\mathcal{V}(\phi))\right). \tag{33}$$

We may get an approximate solution to these equations by following the process laid out in [11]. To start we integrate (32) to find

$$(a(\xi_N)^n\phi'(\xi_N)) = \int_0^{\xi_N} \mathrm{d}\xi a(\xi)^n\frac{\partial\mathcal{V}}{\partial\phi}, \tag{34}$$

then we make the approximation that at ξ_N the constraint equation (23) yields,

$$a'(\xi_N) \simeq -\frac{a(\xi_N)\phi'(\xi_N)}{\sqrt{n(n-1)}} \tag{35}$$

The action is then found from (33) by taking the scalar field to be the constant $\phi_0 = \phi(\xi = 0)$,

$$S_E \simeq \mathrm{V}ol(\mathrm{S^n})\left(-\frac{2\mathcal{V}(\phi_0)}{\mathrm{n}-1}+\left(\frac{\mathrm{n}}{(\mathrm{n}-1)}\right)^{1/2}\mathcal{V}_{,\phi_0}\right)\int \mathrm{d}\xi \mathrm{a}(\xi)^{\mathrm{n}}, \tag{36}$$

The next approximation is that $a(\xi) \simeq H^{-1}\sin(H\xi)$, where $H^2 = \frac{2\mathcal{V}(\phi_0)}{n(n-1)}$. This then leaves us with,

$$S_E \simeq \left(-\frac{2\mathcal{V}(\phi_0)}{n-1}+\left(\frac{n}{(n-1)}\right)^{1/2}\mathcal{V}_{,\phi_0}\right)\frac{I_n}{H^{(n+1)}}Vol(\mathrm{S^n}), \tag{37}$$

$$I_n = \begin{cases} \frac{2^n[((n-1)/2)!]^2}{n!} & \text{if n is odd} \\ \frac{(n-1)!\pi}{2^{(n-1)}(n/2-1)!(n/2)!} & \text{if n is even.} \end{cases} \tag{38}$$

For example we find, $I_2 = \pi/2,\ I_3 = 4/3,\ I_4 = 3\pi/8,\ I_5 = 16/15$. For the archetypal harmonic potential, $\mathcal{V} = \frac{1}{2}\phi^2$, we obtain an estimate for the location of the minima to be $(\phi_0)_{\min} \simeq n\sqrt{\frac{n}{(n-1)}}$, which is approximately linear in n. The corresponding value for the action is

$$S_{\min}(n) \simeq -I_n\left[\frac{n-1}{n}\right]^{(n-1)} Vol(\mathrm{S^n}). \tag{39}$$

The full numerical solutions to (31)-(33) are given in Fig. 3, where the reduced action is found for a range of dimensions. The behaviour is well explained by the approximation scheme, which describes the positions of the minima increasing as n increases, along with the value of the reduced action which also increases with n.

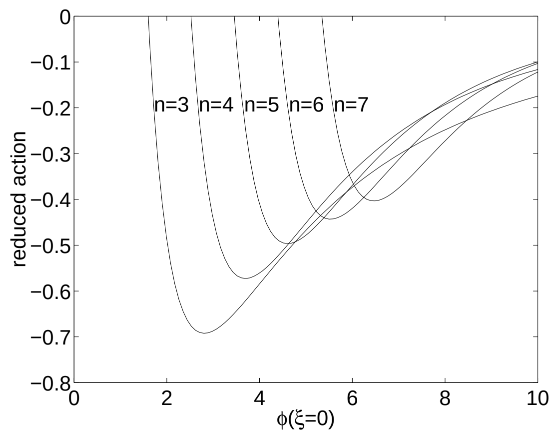

FIG. 3. Numerical results for Hawking Turok instantons of different dimensionalities, $\mathrm{ds}^2 = \mathrm{d}\xi^2 + \mathrm{a}(\xi)^2\mathrm{d}\Omega_{\mathrm{n}}^2$

We are now in a position to compare the total action we found for the product ansatz, -11.9, and this more symmetric case. The example we found before, Fig. 2, was for a six dimensional manifold. the total action of the Hawking-Turok solution in six dimensions is seen from Fig. 3 as $\pi^3(-0.495) = -15.3$, which is lower. We have checked this for a number of dimensions, using $\Lambda_a,\ \Lambda_b > 0$, and found that although a minima existed it had a higher action than the corresponding Hawking-Turok instanton. Although this is not a proof that the lowest action will not be of the product form, we see no reason to suspect otherwise.

Now let us proceed to investigate analytically the behaviour of the action for a more general class of our instantons to see if we can understand why a minima appears at all. Consider eqn (23). If we take p and q of (29) to both be less than one then, using our knowledge of the asymptotic behaviour of the field and scale factors, we find that near a singular pole this equation becomes,

$$\frac{1}{2}n_a(n_a-1)\alpha'^2 + \frac{1}{2}n_b(n_b-1)\beta'^2 + n_a n_b\alpha'\beta' = \frac{1}{2}\phi'^2. \tag{40}$$

Let us consider the case $n_a, n_b >> 1$. Then if we multiply eqn. (40) through by $a^{2n_a}b^{2n_b}$ we find that the left hand side becomes approximately equal to the boundary term in the action squared. Using this observation we can

once more follow through the analysis of Hawking and Turok but this time the analysis will apply to the case of 2 submanifolds. The analysis here is the same as that which lead to (36), except that we must take n_a, $n_b >> 1$ in the constraint equation to find

$$S_E = \left(\mathcal{V}_{,\phi_0} - \frac{2\mathcal{V}(\phi_0)}{(n_a + n_b - 1)}\right) V_{(1)} V_{(2)} \int \mathrm{d}\xi \mathrm{a}(\xi)^{\mathrm{n_a}} \mathrm{b}(\xi)^{\mathrm{n_b}} \tag{41}$$

The next step is to make the approximation that $a(\xi) \simeq H^{-1}\sin(H\xi)$ and $b(\xi) = b(\xi = 0)$ =constant. The end result is that the action is approximated by the value obtained in (37) multiplied by $b(\xi = 0)^{n_b}$ and an extra submanifold volume factor. This then explains why a valley is found in Fig. 2 which is parallel to the $b(0)$ axis, getting deeper as $b(0)$ increases. Clearly the valley cannot keep getting arbitrarily deep, so at some point these approximations break down. We find that the weak link in our chain of reasoning is the assumption that we make about the behaviour of the scale factors as they approach the north pole. For small $b(0)$ then we find that $a(\xi)$ decreases to zero at ξ_N, and our approximation works. When $b(0)$ is larger than some critical value then instead of decreasing to zero, $a(\xi)$ increases and diverges at ξ_N and the approximation of treating it as a sine breaks down.

The important point to take away from all this is that some of our types of instanton with a 'warp product space' topology have local minima of action in parameter space, but the corresponding Hawking Turok instanton (with appropriate dimensionality) will still dominate over them in the semi-classical approximation to the Hartle Hawking wavefunction.

A. A conjecture.

We have seen that the metric associated with the scale factor which vanishes at ξ_S must have $\Lambda_1 > 0$, (27). The same constraint does not apply to the other manifolds. If we were to allow $\Lambda_{i>1} \leq 0$ then Bishop's theorem does not put a limit on the total volume of the $\mathcal{M}_{(i>1)}$; so if the reduced action had a minima with a negative value then (21) could be made arbitrarily negative by increasing $V_{(i>1)}$. We would therefore expect that *negative values of the reduced action occur only if* $\Lambda_{(i)} > 0$ *for all* i. We have checked this for the case of two submanifolds of various dimensions, always confirming this conjecture.

IV. ANALYTICAL SOLUTIONS AND ANALYTICAL CONTINUATION.

The equations are simplified considerably if we actually drop the scalar field ϕ, and replace its potential with a cosmological constant Λ. We then obtain the following analytical solutions [7]. The first is given by,

$$a_{(1)} = \frac{1}{\sqrt{(n_{(1)} - 1)}\,\chi} sin(\chi\xi) \tag{42}$$

$$n_{(1)}(n_{(1)} - 1)\chi^2 = 2\Lambda - \sum_{i>1}\left(\frac{n_{(i)}\Lambda_{(i)}}{a^2_{(i)}}\right) \tag{43}$$

and for $i > 1$,

$$a_{(i)} = \sqrt{\frac{\Lambda_{(i)}}{n_{(1)}\chi^2}}, \tag{44}$$

where $n_{(1)} > 1$. There is a similar solution when $n_{(1)} = 1$. It is also possible that χ may be taken as imaginary, in which case the trigonometric function become hyperbolic, rendering the instanton non-compact. One can still create finite size instantons in this case by introducing domain walls at some value ξ_W. This creates a discontinuity in the gradient of $a_{(1)}$ causing the scale factor to decrease, if the wall tension is large enough. We can see that the limiting behaviour $a_{(1)}(\xi \to 0) \to 1/\sqrt{(n_{(1)} - 1)}$ is consistent with (27) for $\Lambda_{(1)} = 1$, meaning that the metric is regular at the end points. Equation (44) also shows that if we have χ real then $\Lambda_{(i>1)} > 0$, and for imaginary χ $\Lambda_{(i>1)} < 0$. Ricci flat submanifolds would mean a vanishing scale factor for that manifold, so in effect they are not present.

The second analytical solution is,

$$a_{(1)} = \frac{1}{\sqrt{(n_{(1)}-1)}\chi} sin(\chi\xi) \tag{45}$$

$$a_{(2)} = \frac{1}{\sqrt{(n_{(2)}-1)}\chi} cos(\chi\xi) \tag{46}$$

$$(2n_{(1)}n_{(2)} + n_{(1)}(n_{(1)}-1) + n_{(2)}(n_{(2)}-1))\chi^2 = 2\Lambda - \sum_{i>2}(\frac{n_{(i)}\Lambda_{(i)}}{a^2_{(i)}}) \tag{47}$$

and for $i > 2$,

$$a_{(i)} = \sqrt{\frac{\Lambda_{(i)}}{(n_{(1)}+n_{(2)})\chi^2}}, \tag{48}$$

where $n_{(1)} > 1$ and $n_{(2)} > 1$ although there is a similar solution when they both are equal to one. As before, if χ is imaginary we require a domain wall to make the instanton compact. This solution also requires $\Lambda_{(1)} = 1$ and $\Lambda_{(2)} = 1$ if the instanton is to close off in a regular manner at the 'north' and 'south' poles.

Although the majority of the instantons considered in this paper do not analytically continue to lorentzian space times where some of the dimensions are compactified we can use these analytical solutions to demonstrate that there are some that do. One subset of the solutions given above is the product of S^4 with S^n. This can be analytically continued to a four dimensional deSitter space with a static S^n as the internal manifold.

It should be noted that the static nature of this internal manifold is not stable to perturbations. This is of course the manifestation in this context of the problem of stabilising moduli fields in cosmology. We do not attempt to resolve this difficulty here. Some recent mechanisms for stabilising moduli fields in cosmology can be found in [19] and [20].

V. CONCLUSIONS

In this paper, we have derived the equations of motion for a specific warp product of general Einstein metrics. The main conclusion we can draw is that instantons which continue to spaces with compact 'extra' dimensions of the form considered here do not have lower action than the corresponding higher-dimensional Hawking-Turok instanton. However, non trivial minima of the action do occur if the Einstein metrics *all* have positive Λ_i. These results are significant: First, they seem to indicate that the symmetry arguments used by Hawking and Turok in their letter can be applied to higher dimensional cases. Secondly, our analysis applies to a wide range of metrics and cosmological scenarios. Our particular comparison of the instantons involved n-dimensional spheres as our internal compact dimensions. Bishop's theorem then implies that these will provide the lowest possible action for such spacetimes with compact internal dimensions, hence our results apply to any Einstein metric – they will always be beaten by the corresponding Hawking-Turok instantons. This result strongly suggests to us that if the initial quantum state of the universe were to be described by the 'Hartle Hawking proposal' then it would be difficult to explain the presence of extra compact dimensions.

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